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WEAKLY ALMOST PERIODIC FUNCTIONALS CARRIED BY HYPERCOSETS(1)

BY

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Abstract. For G a compact group and H a closed normal subgroup, we show that a weakly almost periodic (w.a.p.) linear functional on the Fourier algebra of G/H lifts to a w.a.p. linear functional on the Fourier algebra of G.

In the case of a compact abelian group G, the dual of a closed subgroup can be identified with a quotient group of the whole dual $\hat{G}$. If G is not abelian, and H is a closed normal subgroup, then an identification space, $\tilde{H}$, of the dual of H, $\hat{H}$, can be identified with a hypercoset structure on $\hat{G}$. Let $H^{\perp}$ be the set of elements of $\hat{G}$ whose kernel contains H. (Recall $\hat{G}$ is the set of equivalence classes of continuous unitary irreducible representations of G.) Then $\tilde{H}$ is identified with the set of hypercosets of $H^{\perp}$, with the trivial representation $H \to \{1\}$ of course identified with $H^{\perp}$ itself. As in the abelian case, the Fourier algebra $A(G)$ of G splits into a direct sum of $A(G/H)$-modules, one for each hypercoset of $H^{\perp}$. Again $A(G/H)$ itself corresponds to $H^{\perp}$. We show here that each of these modules is finitely generated, and use this result to show that weakly almost periodic (w.a.p.) linear functionals on $A(G/H)$ lift to w.a.p. linear functionals on $A(G)$ (the set of such is denoted $W(\hat{G})$).

We show that if G has an infinite abelian homomorphic image, then the space of Fourier-Stieltjes transforms of measures on G is not dense in $W(\hat{G})$, and $W(\hat{G})$ is not equal to $\mathscr{L}^{\infty}(\hat{G})$, the dual of $A(G)$. We will use some of the methods developed in our previous paper on w.a.p. functionals [6].

1. **Notation and hypercosets.** Let G be a compact nonabelian group. Using our previous notation [3, Chapters 7, 8] we let $\hat{G}$ denote the set of equivalence classes of continuous unitary irreducible representations of G. For $\alpha \in \hat{G}$, choose $T_\alpha \in \alpha$, then T_α is a continuous homomorphism of G into $U(n_\alpha)$, the group of $n_\alpha \times n_\alpha$ unitary matrices where n_α is the dimension of α. We use $T_\alpha(x)_{ij}$ to denote the matrix entries of $T_\alpha(x)$, $1 \leq i, j \leq n_\alpha$, and $T_{\alpha ij}$ to denote the (continuous) function $x \mapsto T_\alpha(x)_{ij}$. Let $V_\alpha = \mathrm{Sp}\,\{T_{\alpha ij} : 1 \leq i, j \leq n_\alpha\}$ (where Sp denotes the linear span), then V_α is an n_α^2-dimensional space of continuous functions invariant under left

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and right translation by G. Further let $\chi_\alpha(x) = \text{trace}(T_\alpha(x)) = \sum_{i=1}^n T_\alpha(x)_{ii}$. The function χ_α is called the character of α and it, as well as V_α, is independent of the choice of $T_\alpha \in \alpha$.

For $\alpha, \beta \in \hat{G}$ one can form the tensor product $T_\alpha \otimes T_\beta$ of the two representations. This tensor product decomposes into irreducible components: $T_\alpha \otimes T_\beta \cong \sum_{\gamma \in \hat{G}} M_{\alpha\beta}(\gamma) T_\gamma$, where $M_{\alpha\beta}(\gamma) = \int_G \chi_\alpha \chi_\beta \bar{\chi}_\gamma \, dm_G$, a nonnegative integer (m_G is the normalized Haar measure on G). This decomposition can also be written in the form $\chi_\alpha \chi_\beta = \sum_\gamma M_{\alpha\beta}(\gamma) \chi_\gamma$ (a finite sum). For $E, F \subset \hat{G}$, we define

$$E \otimes F = \{\gamma \in \hat{G} : M_{\alpha\beta}(\gamma) \neq 0, \text{ some } \alpha \in E, \beta \in F\}.$$

This operation makes $\hat{G}$ into a hypergroup. For each $\alpha \in \hat{G}$, there is a conjugate $\bar{\alpha} \in \hat{G}$ such that $\chi_{\bar{\alpha}}(x) = (\chi_\alpha(x))^-$ $(x \in G)$. If $E \subset \hat{G}$ and $E \otimes E \subset E$, then E is called a subhypergroup, and if further $\bar{E} = \{\bar{\alpha} : \alpha \in E\} \subset E$ then E is called a normal subhypergroup.

For any set $S \subset G$, let $S^\perp = \{\alpha \in \hat{G} : S \subset \text{kernel } T_\alpha\}$ then $S^\perp$ is a normal subhypergroup. For $E \subset \hat{G}$, let $E^\perp = \bigcap_{\alpha \in E} (\text{kernel } T_\alpha)$, a closed normal subgroup of G. Helgason [7] has shown that if H is a closed normal subgroup of G then $(H^\perp)^\perp = H$.

If E is a normal subhypergroup of $\hat{G}$ and $\alpha \in \hat{G}$ then $\alpha \otimes E$ is called a hypercoset of E. We will prove later that $\hat{G}$ is the disjoint union of hypercosets of E.

Let X be an n-dimensional complex inner product space. Let $\mathscr{B}(X)$ be the space of linear maps: $X \to X$. The operator norm of $A \in \mathscr{B}(X)$ is defined to be $\|A\|_\infty = \sup\{|A\xi| : \xi \in X, |\xi| \leq 1\}$. The trace of A is defined to be $\text{Tr } A = \sum_{i=1}^n (A\xi_i, \xi_i)$ where $\{\xi_i\}_{i=1}^n$ is any orthonormal basis for X and $(\cdot, \cdot)$ is the inner product in X. We define the dual norm to $\|\cdot\|_\infty$ by

$$\|A\|_1 = \sup\{|\text{Tr}(AB)| : B \in \mathscr{B}(X), \|B\|_\infty \leq 1\}.$$

One can show that $\|A\|_1 = \text{Tr}(|A|)$, where $|A| = (A^*A)^{1/2}$.

Let ϕ be a set $\{\phi_\alpha : \alpha \in \hat{G}, \phi_\alpha \in \mathscr{B}(C^{n_\alpha}), \sup_\alpha \|\phi_\alpha\|_\infty < \infty\}$. The set of all such ϕ is denoted by $\mathscr{L}^\infty(\hat{G})$. It is a C^*-algebra under the norm $\|\phi\|_\infty = \sup\{\|\phi_\alpha\|_\infty : \alpha \in \hat{G}\}$ and coordinatewise operations (* denotes the operator adjoint).

Let $\mathscr{L}^1(\hat{G}) = \{\phi \in \mathscr{L}^\infty(\hat{G}) : \|\phi\|_1 = \sum_{\alpha \in \hat{G}} n_\alpha \|\phi_\alpha\|_1 < \infty\}$. Then $\mathscr{L}^1(\hat{G})$ with the norm $\|\cdot\|_1$ is a Banach space and its dual may be identified with $\mathscr{L}^\infty(\hat{G})$ under the pairing $\langle \phi, \psi \rangle = \sum_\alpha n_\alpha \text{Tr}(\phi_\alpha \psi_\alpha)$ $(\phi \in \mathscr{L}^1(\hat{G}), \psi \in \mathscr{L}^\infty(\hat{G}))$. Let $\mu \in M(G)$, the measure algebra of G, then the Fourier transform of μ, $\hat{\mu}$, is the function $\alpha \mapsto \hat{\mu}_\alpha = \int_G T_\alpha(x^{-1}) \, d\mu(x)$ $(\alpha \in \hat{G})$, and $\hat{\mu} \in \mathscr{L}^\infty(\hat{G})$ with $\|\hat{\mu}\|_\infty \leq \|\mu\|$. If $f \in C(G)$ (the continuous functions on G), then $\hat{f}_\alpha = \int_G T_\alpha(x^{-1}) f(x) \, dm_G(x)$ $(\alpha \in \hat{G})$.

We will now define $A(G)$, the Fourier algebra of G. Let

$$A(G) = \{f \in C(G) : \hat{f} \in \mathscr{L}^1(\hat{G})\},$$

then $A(G)$ is in fact isomorphic to $\mathscr{L}^1(\hat{G})$, since for $\phi \in \mathscr{L}^1(\hat{G})$ the function $f(x) = \sum_{\alpha \in \hat{G}} n_\alpha \text{Tr}(\phi_\alpha T_\alpha(x))$ $(x \in G)$ is continuous and $\hat{f} = \phi$. Put $\|f\|_A = \|\hat{f}\|_1$. Further

$A(G)$ is a (commutative) Banach algebra under the pointwise operations on G; and its dual is $\mathscr{L}^\infty(\hat{G})$, under the pairing

$$\langle f, \phi\rangle = \sum_\alpha n_\alpha \operatorname{Tr}(\hat{f}_\alpha \phi_\alpha) \qquad (f \in A(G), \phi \in \mathscr{L}^\infty(\hat{G}));$$

for proofs see [3, p. 93].

DEFINITION 1.1. For $\phi \in \mathscr{L}^\infty(\hat{G})$ define the carrier of ϕ, $\operatorname{cr}\phi = \{\alpha \in \hat{G} : \phi_\alpha \neq 0\}$. For $E \subset \hat{G}$, let $\mathscr{L}^\infty(E) = \{\phi \in \mathscr{L}^\infty(\hat{G}) : \operatorname{cr}\phi \subset E\}$, and let $A(E) = \{f \in A(G) : \operatorname{cr}\hat{f} \subset E\}$. In fact $A(E)$ is the closed span of $\{V_\alpha : \alpha \in E\}$.

PROPOSITION 1.2. *The spaces $A(E)$, $E \subset \hat{G}$, are exactly the closed subspaces of $A(G)$ which are invariant under left and right translation by G. The dual of $A(E)$ is $\mathscr{L}^\infty(E)$.*

PROPOSITION 1.3. *Let $E, F \subset \hat{G}$, then the closed linear span of*

$$\{fg : f \in A(E), g \in A(F)\}$$

is equal to $A(E \otimes F)$.

COROLLARY 1.4. *For $E \subset \hat{G}$, $A(E)$ is a subalgebra of $A(G)$ if and only if E is a subhypergroup. Further $A(E)$ is a conjugate-closed ($f \mapsto \bar{f}$) subalgebra of $A(G)$ if and only if E is a normal subhypergroup, and in that case*

$$A(E) = \{f \in A(G) : f(h_1 x h_2) = f(x), \text{ for all } h_1, h_2 \in E^\perp, x \in G\},$$

the functions in $A(G)$ constant on cosets of $E^\perp$, a closed normal subgroup of G.

COROLLARY 1.5. *If E is a finite subhypergroup of $\hat{G}$ then E is normal.*

Proof. In fact $A(E)$ is a finite dimensional subalgebra of a conjugate closed algebra $A(G)$ and is thus itself conjugate-closed (since the maximal ideal space of $A(E)$ is a finite set). □

REMARK 1.6. The Fourier algebra of a compact group G is the subject of [3, Chapter 8]. Helgason [7] constructed the duality between normal subhypergroups of $\hat{G}$ and closed normal subgroups of G. Translation-invariant uniformly closed linear subspaces of $C(G)$ are discussed by Rider in [9]. What is observed by Iltis [8], that finite subhypergroups are normal, is implicit in Rider [9, p. 980].

2. **Restrictions of representations to normal subgroups.** In this section, G denotes a compact group, and H denotes a closed normal subgroup of G. Define $\hat{H}$ similarly to $\hat{G}$, and denote the character of $\gamma \in \hat{H}$ by ξ_γ, and let $T_\gamma^H \in \gamma$. Denote the normalized Haar measure of H by m_H. There exists a natural homomorphism of G into the group of automorphisms of H; namely, for $x \in G$, let $S_x h = xhx^{-1}$ $(h \in H)$, then S_x is an automorphism of H. Each S_x induces a permutation $\hat{x}$ on $\hat{H}$ such that $\xi_{\hat{x}\gamma}(h) = \xi_\gamma(S_x h)$ $(h \in H)$. Now define an equivalence relation on $\hat{H}$ by $\gamma_1 \sim \gamma_2$ if and only if $\gamma_2 = \hat{x}\gamma_1$ for some $x \in G$ $(\gamma_1, \gamma_2 \in \hat{H})$. Denote the set of such equivalence classes by $\tilde{H}$. Let $\alpha \in \hat{G}$ then $T_\alpha | H$ is a continuous unitary representation of H and thus decomposes:

$$T_\alpha | H = \sum_{\gamma \in \hat{H}} a_\gamma T_\gamma^H$$

where the a_γ's are nonnegative integers and only finitely many are nonzero.

THEOREM 2.1. *Let $\alpha \in \hat{G}$, then there is a positive integer N_α and a class $\Gamma_\alpha \in \tilde{H}$ such that*

$$\chi_\alpha | H = N_\alpha \sum_{\gamma\in\Gamma_\alpha} \xi_\gamma.$$

Further each equivalence class in $\tilde{H}$ is finite.

Proof. This theorem is nothing but the compact group analogue to the well-known finite groups result (see e.g. [1, p. 278]). We sketch an argument. Write $\chi_\alpha | H = \sum_\gamma c_\gamma \xi_\gamma$. Now $\chi_\alpha | H$ is invariant under each S_x, $x \in G$, thus if $\gamma_1 \sim \gamma_2$ then $c_{\gamma_1} = c_{\gamma_2}$. It remains to show that if $c_{\gamma_1} \neq 0$ and $c_{\gamma_2} \neq 0$ then $\gamma_1 \sim \gamma_2$. Now $d\mu = \sum_{\gamma\in\Gamma_\alpha} \xi_\gamma \, dm_H$ is a central measure in $M(G)$, and for any $\alpha \in \hat{G}$, μ is orthogonal to either all or none of the diagonal entry functions. If $\gamma' \notin \Gamma_\alpha$, then

$$\int_H \xi_{\gamma'} \left(\sum_{\gamma\in\Gamma_\alpha} \xi_\gamma \right) dm_H = 0.$$

Thus the class $\Gamma_\alpha \in \tilde{H}$ is uniquely determined and is evidently finite. However any $\gamma \in \hat{H}$ appears in the restriction of some $\alpha \in \hat{G}$ (induced representation argument) and thus any class in $\tilde{H}$ is finite. □

COROLLARY 2.2. *For any $\alpha, \beta \in \hat{G}$, either $n_\beta \chi_\alpha | H = n_\alpha \chi_\beta | H$ or $\int_H \chi_\alpha \bar{\chi}_\beta \, dm_H = 0$.*

Proof. For $\alpha, \beta \in \hat{G}$, if $\Gamma_\alpha = \Gamma_\beta$ then $n_\alpha / N_\alpha = \sum_{\gamma\in\Gamma_\alpha} \xi_\gamma(e) = n_\beta / N_\beta$ (e is identity in G). If $\Gamma_\alpha \neq \Gamma_\beta$, then $\int_H \chi_\alpha \bar{\chi}_\beta \, dm_H = 0$ by the orthogonality relations for characters of H. □

REMARK 2.3. Rider uses this corollary in [10].

REMARK 2.4. For $\alpha \in \hat{G}$, $\alpha \in H^\perp$ if and only if $\Gamma_\alpha = \{\{1\}\}$ ($\{1\}$ denotes the trivial representation $H \to \{1\}$), and in this case, $\chi_\alpha | H = n_\alpha = N_\alpha$.

THEOREM 2.5. *For $\alpha, \beta \in \hat{G}$, $\alpha \in \beta \otimes H^\perp$ if and only if $\Gamma_\alpha = \Gamma_\beta$. Thus $\hat{G}$ is split into disjoint hypercosets, and these hypercosets are indexed by $\tilde{H}$.*

Proof. Let $\alpha \in \beta \otimes H^\perp$, then there exists some $\delta \in H^\perp$ such that $M_{\beta\delta}(\alpha) \neq 0$, that is $\chi_\beta \chi_\delta = \chi_\alpha + \phi$, where ϕ is some nonnegative integer combination of characters. Now restrict to H to obtain

$$n_\delta N_\beta \sum_{\gamma\in\Gamma_\beta} \xi_\gamma = N_\alpha \sum_{\gamma\in\Gamma_\alpha} \xi_\gamma + \phi | H,$$

thus $\Gamma_\alpha \subset \Gamma_\beta$, hence $\Gamma_\alpha = \Gamma_\beta$.

Conversely if $\Gamma_\alpha = \Gamma_\beta$, then $(\chi_\alpha | H)(\chi_\beta | H)^- = 1 + \phi$ (some ϕ as above). (This follows from the relation $M_{\gamma\bar{\gamma}}(\{1\}) = \int_H |\xi_\gamma|^2 \, dm_H = 1$ for any $\gamma \in \hat{H}$.) This implies that $(\alpha \otimes \bar{\beta}) \cap H^\perp \neq \varnothing$. Thus there is a $\delta \in H^\perp$ such that $M_{\alpha\bar{\beta}}(\delta) \neq 0$, but

$$M_{\alpha\bar{\beta}}(\delta) = \int_G \chi_\alpha \bar{\chi}_\beta \bar{\chi}_\delta \, dm_G = \int_G \bar{\chi}_\alpha \chi_\beta \chi_\delta \, dm_G = M_{\beta\delta}(\alpha)$$

and so $\alpha \in \beta \otimes H^\perp$. □

Observe that the Fourier algebra of G/H is isomorphic to a closed subalgebra of $A(G)$, namely $A(H^{\perp})$ (which we will denote by A_H), since $(G/H)^{\wedge}$ may be identified with $H^{\perp}$. We will now decompose $A(G)$ into a direct sum of A_H-modules.

THEOREM 2.6. *To each $\Gamma \in \tilde{H}$ there corresponds a closed subspace A_Γ of $A(G)$ which is also an A_H-module. Further each $f \in A(G)$ has a unique decomposition $f = \sum_{\Gamma \in \tilde{H}} f_\Gamma$, where $f_\Gamma \in A_\Gamma$ and $\|f\|_A = \sum_{\Gamma \in \tilde{H}} \|f_\Gamma\|_A$. Also the A_Γ's are the minimal closed left and right translation invariant A_H-submodules of $A(G)$.*

Proof. For $\Gamma \in \tilde{H}$, let $E_\Gamma = \{\alpha \in \hat{G} : \Gamma_\alpha = \Gamma\}$, that is, the hypercoset of H corresponding to Γ. Then put $A_\Gamma = A(E_\Gamma)$. Clearly $\hat{G}$ is the disjoint union of $\{E_\Gamma\}_{\Gamma \in \tilde{H}}$, so the decomposition of $A(G)$ follows from the obvious decomposition of $\mathscr{L}^1(\hat{G})$. Let $\Gamma \in \tilde{H}$ and choose $\alpha \in E_\Gamma$ then $E_\Gamma = \alpha \otimes H^{\perp}$, so that $E_\Gamma \otimes H^{\perp} = E_\Gamma$ and thus, by Proposition 1.3, $A_H \cdot A_\Gamma \subset A_\Gamma$. So A_Γ is a closed A_H-submodule of $A(G)$. If a nontrivial closed left and right translation invariant A_H-module is contained in A_Γ, then it is determined by some nonempty subset $F \subset E_\Gamma$. But if $\alpha \in F$ and $\beta \in H^{\perp}$ then $\alpha \otimes \beta \subset F$, thus F is a hypercoset, hence equals E_Γ. □

REMARK 2.7. If G is abelian then each A_Γ has a single generator (in the algebraic as well as the topological sense) over A_H. In the general case for $\Gamma \in \tilde{H}$ and some $\alpha \in E_\Gamma$, the functions $\{T_{\alpha ij} : 1 \leqq i, j \leqq n_\alpha\}$ generate A_Γ topologically, but it is not clear that they do so algebraically. However the following is true.

THEOREM 2.8. *Let $\Gamma \in \tilde{H}$, then A_Γ is a finitely generated A_H-module, that is there exists $g_1, \ldots, g_m \in A_\Gamma$ (some $m < \infty$) so that each $f \in A_\Gamma$ may be written as $f = \sum_{i=1}^m k_i g_i$, with $k_i \in A_H$. Further there exists a constant $M < \infty$, such that the functions k_i may be chosen with $\|k_i\|_A \leqq M \|f\|_A$.*

Proof. In a paper of Dunkl [2] the following is shown: let τ be a continuous unitary representation of H on a finite dimensional space V, and let $A(G, V)$ be the space of V-valued functions on G with each coordinate function in $A(G)$. Define

$$M(\tau) = \{f \in A(G, V); f(hx) = \tau(h)f(x) \text{ for all } h \in H,\ x \in G\},$$

denoted $A(\tau)$ in [2]. Then $M(\tau)$ is a finitely generated (algebraically) A_H-module.

We now point out the applicability of this theorem to the present situation. Pick $\alpha \in \Gamma$, and let $V = V_\alpha | H$. Recall $V_\alpha = \mathrm{Sp}\{T_{\alpha ij} : 1 \leqq i, j \leqq n_\alpha\}$ so that V is a finite dimensional space of continuous functions on H, and is in fact the left and right translation invariant (by H) space generated by $\{\xi_\gamma : \gamma \in \Gamma\}$. This shows that V depends only on Γ, that for any $f \in A_\Gamma$, $f|H \in V$, and finally that V is invariant under each S_x, $x \in G$ (that is, if $g \in V$, $x \in G$, then the function $h \mapsto g(xhx^{-1})$ is in V $(h \in H)$). Observe that a continuous unitary representation τ of H is realized on V, namely right translation, with the inner product on V given by $(f, g)_H = \int_H f\bar{g}\, dm_H$ $(f, g \in V)$, and $\tau(h)f(h_1) = f(h_1 h)$ $(f \in V,\ h, h_1 \in H)$.

We claim that $M(\tau) = A_\Gamma$, in fact if $f \in A_\Gamma$ then assign to each $x \in G$ the function $f(x, \cdot) : h_1 \mapsto f(h_1 x) = (R(x)f)(h_1)$. Now A_Γ is invariant under the right translation

$R(x)$ so $R(x)f|H \in V$, thus $f(x, \cdot) \in V$. Further for $x \in G$, $h \in H$, $f(hx, h_1) = f(h_1hx) = f(x, h_1h) = \tau(h)f(x, h_1)$ $(h_1 \in H)$, that is, $f(hx, \cdot) = \tau(h)f(x, \cdot)$. Finally to check the coordinate functions of $f(x, \cdot)$ let $g \in V$ and consider the function $x \mapsto (f(x, \cdot), g)_H = \int_H f(hx)(g(h))^- \, dm_H = \mu * f(x)$, where μ is the measure $(g(h^{-1}))^- \, dm_H(h)$, and so $\mu * f \in A(G)$. Conversely, if $f \in M(\tau)$, so f is of the form $f(x, h)$, with $f(x, \cdot) \in V$, then put $f(x) = f(x, e)$. Thus $f \in A(G)$ (by finite dimensionality of V, point evaluation is a bounded linear functional). Further for each $x \in G$ let $g = R(x)f|H$, then $g(h) = f(hx) = f(hx, e) = \tau(h)f(x, e) = f(x, h)$ so the function $g \in V$, thus $f \in A_\Gamma$. Hence $A_\Gamma = M(\tau)$ and thus there exist generators $g_1, \ldots, g_m \in A_\Gamma$ (some $m < \infty$).

Now consider the bounded linear map $T: A_H \times A_H \times \cdots \times A_H$ (m copies) $\to A_\Gamma$ defined by $T(k_1, \ldots, k_m) = \sum_{i=1}^m k_i g_i$. By the above paragraph T is onto and so by the open mapping theorem there exists $M < \infty$ such that $\{T(k_1, \ldots, k_m) : \|k_j\|_A \leqq M\} \supset \{f \in A_\Gamma : \|f\|_A \leqq 1\}$. $\square$

3. **Homomorphisms.** Let π be a continuous homomorphism of a compact group G into a compact group K, and let H be the kernel of π. Then π induces the map $\pi_1: C(K) \to C(G)$, given by $\pi_1 f(x) = f(\pi x)$, $f \in C(K)$, $x \in G$. The adjoint of π_1, denoted by π^*, takes $M(G)$ into $M(K)$. Further π_1 maps $A(K)$ into $A(G)$, since $A(K)$ is spanned by the continuous positive definite functions and these are preserved by π_1. Also $\pi_1|A(K)$ is a bounded operator on $A(K)$ since each $f \in A(K)$ is a sum $f = f_1 - f_2 + i(f_3 - f_4)$, f_i positive definite and $\sum_{i=1}^4 f_i(e) \leqq 2\|f\|_A$. Finally the adjoint of $\pi_1|A(K)$ is a bounded map $\hat{\pi} : \mathscr{L}^\infty(\hat{G})$ into $\mathscr{L}^\infty(\hat{K})$. Let $\mathscr{M}(\hat{G})$, $\mathscr{M}(\hat{K})$ be the closures of $M(G)^\wedge$, $M(K)^\wedge$ in $\mathscr{L}^\infty(\hat{G})$ and $\mathscr{L}^\infty(\hat{K})$ respectively.

PROPOSITION 3.1. $\hat{\pi}\mathscr{M}(\hat{G}) \subset \mathscr{M}(\hat{K})$.

Proof. Let $\mu \in M(G)$, then $\hat{\mu}$ satisfies the following: $\langle f, \hat{\mu} \rangle = \int_G f(x^{-1}) \, d\mu(x)$, $f \in A(G)$. Now let $g \in A(K)$, then

$$\langle g, \hat{\pi}\hat{\mu} \rangle = \langle \pi_1 g, \hat{\mu} \rangle = \int_G g(\pi x^{-1}) \, d\mu(x) = \int_K g(k^{-1}) \, d\pi^*\mu(k) = \langle g, (\pi^*\mu)^\wedge \rangle.$$

Thus $\hat{\pi}\hat{\mu} = (\pi^*\mu)^\wedge \in M(K)^\wedge$. The continuity of $\hat{\pi}$ finishes the proof. $\square$

Observe that π factors into $G \to G/H \to K$, where G/H is identified with a closed subgroup of K. Further $M(G/H)$ is identified with a closed subalgebra of $M(G)$, namely $m_H * M(G)$ (note m_H is an idempotent, see [3, Chapter 9]). Also $\mathscr{L}^\infty((G/H)^\wedge) \cong \mathscr{L}^\infty(H^\perp)$ and $\mathscr{M}((G/H)^\wedge) \cong \mathscr{M}(\hat{G}) \cap \mathscr{L}^\infty(H^\perp)$ (since $\hat{m}_H$ is the projection of $\mathscr{L}^\infty(\hat{G})$ onto $\mathscr{L}^\infty(H^\perp)$).

Finally $\hat{\pi}$ takes $\mathscr{M}(\hat{G})$ onto $\mathscr{M}(\hat{K})$, or $\mathscr{L}^\infty(\hat{G})$ onto $\mathscr{L}^\infty(\hat{K})$ if and only if π maps G onto K, for otherwise πG is a proper closed subgroup of K, and $\phi \in \hat{\pi}\mathscr{L}^\infty(\hat{G})$ if and only if spt $\phi \subset \pi G$ (where the support of ϕ, spt ϕ, is the least compact subset $E \subset K$ with the property that $f \in A(K)$, $f = 0$ on a neighborhood of E implies $\langle f, \phi \rangle = 0$).

Now we investigate the effect of $\hat{\pi}$ on $W(\hat{G})$, the weakly almost periodic (w.a.p.) elements of $\mathscr{L}^\infty(\hat{G})$. We state some appropriate definitions and results from our previous paper [6].

PROPOSITION 3.2. *$\mathscr{L}^\infty(\hat{G})$ is an $A(G)$-module. The action is defined by $\langle g, f\cdot\phi\rangle = \langle fg, \phi\rangle$ $(f, g \in A(G), \phi \in \mathscr{L}^\infty(\hat{G}))$, and $\|f\cdot\phi\|_\infty \leqq \|f\|_A\|\phi\|_\infty$. Further* $\operatorname{cr}(f\cdot\phi) \subset (\operatorname{cr} f)^- \otimes \operatorname{cr}\phi$.

DEFINITION 3.3. For $\phi \in \mathscr{L}^\infty(\hat{G})$, one says that ϕ is weakly almost periodic if the map $f \mapsto f\cdot\phi$ is a weakly compact operator of $A(G)$ into $\mathscr{L}^\infty(\hat{G})$ $(f \in A(G))$. The set of all such ϕ is denoted by $W(\hat{G})$.

DEFINITION 3.4. Let $B=\{f \in A(G) : \|f\|_A \leqq 1\}$. For $\alpha \in \hat{G}$, let $B_\alpha = B \cap V_\alpha$. Let $E=\bigcup_{\alpha\in\hat{G}} B_\alpha$.

Some properties of $W(\hat{G})$ (see [6]):

(1) $W(\hat{G})$ is a closed submodule of $\mathscr{L}^\infty(\hat{G})$.

(2) For $\phi \in \mathscr{L}^\infty(\hat{G})$ to be in $W(\hat{G})$ it is necessary and sufficient that $\{f_n\cdot\phi\}$ have a weakly convergent subsequence for any sequence $\{f_n\} \subset E$ (also true if E is replaced by B).

THEOREM 3.5. *Let π be a continuous homomorphism of G into K (compact groups). Then $\hat{\pi}W(\hat{G}) \subset W(\hat{K})$.*

Proof. Suppose $\phi \in W(\hat{G})$, and $\{f_n\}$ is a bounded sequence in $A(K)$. Then $\{\pi_1 f_n\}$ is a bounded sequence in $A(G)$, and there exists a subsequence such that $(\pi_1 f_{n_j})\cdot\phi$ converges weakly to $\psi \in \mathscr{L}^\infty(\hat{G})$. But $\hat{\pi}((\pi_1 f_{n_j})\cdot\phi) = f_{n_j}\cdot(\hat{\pi}\phi)$, so $f_{n_j}\cdot\hat{\pi}\phi$ converges weakly to $\hat{\pi}\psi \in \mathscr{L}^\infty(\hat{K})$ (for $\hat{\pi}$, being strongly continuous, is weakly continuous). Hence $\hat{\pi}\phi \in W(\hat{K})$. □

Henceforth we assume π is onto K so we identify $\hat{K}$ with $H^\perp$, and $\mathscr{L}^\infty(\hat{K})$ with $\mathscr{L}^\infty(H^\perp)$. We have just seen that the restriction map $\hat{\pi}$: $\mathscr{L}^\infty(\hat{G}) \to \mathscr{L}^\infty(H^\perp)$ takes $W(\hat{G})$ into $W(\hat{K})$. We will now show that in fact $\hat{\pi}(W(\hat{G}) \cap \mathscr{L}^\infty(H^\perp)) = W(\hat{K})$.

DEFINITION 3.6. Let $\{\phi_n\}$ be a sequence in $\mathscr{L}^\infty(\hat{G})$. Say $\phi_n \xrightarrow{n} \phi \in \mathscr{L}^\infty(\hat{G})$ quasi-uniformly if $(\phi_n)_\alpha \xrightarrow{n} \phi_\alpha$ for each $\alpha \in \hat{G}$, and for each $\varepsilon > 0$, $N = 1, 2, 3, \ldots$, there exist integers $m_1, \ldots, m_k \geqq N$, such that $\min_{1\leqq i\leqq k} \|(\phi_{m_i})_\alpha - \phi_\alpha\|_\infty < \varepsilon$ for each $\alpha \in \hat{G}$.

THEOREM 3.7 [6]. *Let $\{\phi_n\} \subset \mathscr{L}^\infty(\hat{G})$. Then $\phi_n \xrightarrow{n} \phi \in \mathscr{L}^\infty(\hat{G})$ weakly if and only if $\sup_n \|\phi_n\|_\infty < \infty$, and every subsequence of $\{\phi_n\}$ converges quasi-uniformly to ϕ.*

THEOREM 3.8. *Let $\phi \in W(\hat{K})$, that is, $\phi \in \mathscr{L}^\infty(H^\perp)$, and for each bounded sequence, $\{f_n\} \subset A(K) = A_H$ (see previous section), $\{f_n\cdot\phi\}$ has a weakly convergent subsequence. Then $\phi \in W(\hat{G})$ (note $\phi_\alpha = 0$ for $\alpha \notin H^\perp$).*

Proof. Let $\{f_n\} \subset E = \bigcup_\alpha B_\alpha$, with $f_n \in B_{\alpha_n}$, $n = 1, 2, 3, \ldots$. We must show that $\{f_n\cdot\phi\}$ has a weakly convergent subsequence. There are two possibilities for $\{\alpha_n\}$:

(1) There are infinitely many distinct cosets $\bar{\alpha}_n \otimes H^\perp$. That is, there exists a subsequence f_{n_j} such that the sets $\operatorname{cr}(f_{n_j}\cdot\phi) \subset \bar{\alpha}_{n_j} \otimes H^\perp$ are all disjoint. Then $f_{n_j}\cdot\phi \xrightarrow{j} 0$ weakly by Theorem 3.7.

(2) Infinitely many $\alpha_n \in \alpha \otimes H^{\perp}$, some $\alpha \in \hat{G}$. Thus there is a bounded subsequence f_{n_j} in A_Γ, where $\Gamma = \Gamma_\alpha$ (recall Theorem 2.6). By Theorem 2.8, there exist $g_1, \ldots, g_m \in A_\Gamma$ and functions $h_{ij} \in A_H$, and $M < \infty$, such that $f_{n_j} = \sum_{i=1}^m h_{ij} g_i$, and $\|h_{ij}\|_A \leqq M$, all i, j. By successively extracting subsequences from $\{h_{1j}\}, \{h_{2j}\}, \ldots, \{h_{mj}\}$ and reindexing, we obtain $\psi_1, \ldots, \psi_m \in \mathscr{L}^\infty(H^\perp)$ such that $h_{ij} \cdot \phi \xrightarrow{j} \psi_i$ weakly, $i = 1, \ldots, m$. The map $\psi \mapsto g_i \cdot \psi$ on $\mathscr{L}^\infty$ is strongly, hence weakly continuous, thus

$$f_{n_j} \cdot \phi = \sum_{i=1}^m g_i \cdot (h_{ij} \cdot \phi) \xrightarrow{j} \sum_{i=1}^m g_i \cdot \psi_i \quad \text{weakly.} \quad \square$$

Corollary 3.9. *If* $W(\hat{K}) \neq \mathscr{L}^\infty(\hat{K})$ *then* $W(\hat{G}) \neq \mathscr{L}^\infty(\hat{G})$. *If* $\mathscr{M}(\hat{K}) \neq W(\hat{K})$ *then* $\mathscr{M}(\hat{G}) \neq W(\hat{G})$. (*Recall from* [6] *that* $\mathscr{M}(\hat{G}) \subset W(\hat{G})$.)

Corollary 3.10. *If G has an infinite abelian image, then* $\mathscr{M}(\hat{G}) \neq W(\hat{G}) \neq \mathscr{L}^\infty(\hat{G})$.

Proof. If K is an infinite compact abelian group, then $\mathscr{M}(\hat{K}) \neq W(\hat{K}) \neq \mathscr{L}^\infty(\hat{K})$ (see [3, Chapter 4] and [6]). $\square$

Remark 3.11. In [4] we show that $W(\hat{G}) \neq \mathscr{L}^\infty(\hat{G})$ for any infinite compact group G. In [5] we show that $\mathscr{M}(\hat{G}) \neq W(\hat{G})$ for any compact group G which contains an infinite abelian subgroup.

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